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REGULAR ARTICLE

Effect of modeling fixed cost in a serial inventory system with periodic review

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Abstract We study a two-echelon serial inventory system with stochastic demand. We assume that fixed ordering costs are charged only when an order initiates a non-zero shipment. The system is centrally controlled and ordering decisions are based on echelon base stock policies. The review period of the upper echelon is an integer multiple of the review period of the lower echelon. We derive an exact analytical expression for the objective function. From this expression, we determine optimal base stock levels and review periods. Through a numerical study we show that there may be several combinations of optimal review periods and that under high fixed ordering costs both stockpoints have the same order frequency. In addition, we identify parameter settings under which the system behaves like a PUSH-system, where the upstream stockpoint never holds any stock. Generally, in literature fixed ordering costs are charged at every review moment, even if no shipment results due to zero upstream stock. We test the impact of this simplifying assumption and illustrate when it is justified.

Keywords Multi-echelon inventory systems · Stochastic demand · Periodic review · Fixed cost

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1 Introduction

Inventory models can be divided into two different classes based on the way inventory levels are monitored. While the class of models with continuous review of the inventory status can be applied very well for slow moving products like spare parts, a different modeling approach should be used for fast moving items, as usually observed in production and distribution networks. In the latter situation decisions are often made on fixed days of the week/month so that multiple activities related to ordering an item, such as order picking and invoicing, can be more easily coordinated. Periodic review also allows to coordinate ordering decisions for multiple items.

The simplest periodic review model for a single stockpoint and a single product with stochastic demand is an (R, S) policy, where R denotes the review interval and S the base stock level. In this paper an inventory model for a single product with stochastic demand is investigated, too, but we study a two-echelon serial inventory model under central control. Stock levels are reviewed periodically and both stockpoints are allowed to have different review periods. Ordering decisions are based on global system information and we further assume that both stockpoints use echelon base stock levels to determine the size of their replenishment orders.

Since the policy structure is given, it remains to determine the policy parameters. The overall goal is, in general, to minimize costs composed of fixed costs related to each order, costs for keeping items on stock and costs for not being able to deliver. We assume linear holding and backorder costs and we additionally suppose that fixed costs are charged whenever an order is executed and material is shipped to a stockpoint. We call these costs fixed ordering costs. These costs may include set-up costs of machines, as well as transportation costs. This means that fixed ordering costs can only be incurred if an order is placed and material is available at the up-stream stockpoint to be shipped. The latter aspect is often ignored and fixed costs are related to each order placed, independent of the state of the on-hand inventory of the supplying stockpoint. The advantage of this approach is a less complicated analysis of the system, but the impact of this simplifying assumption on the optimal solution is not clear and the ordering costs are overestimated.

The research presented in this paper is motivated by two questions: (1) How can the optimal policy parameters for the policy described above be computed under the given cost assumptions? (2) Does the simplifying assumption for the fixed ordering cost have an impact on the optimal solution?

The contribution of our paper to existing literature consists of the following aspects. First, we derive exact expressions for the long-run average cost taking into account that fixed ordering cost are incurred when a positive shipment quantity is initiated. Second, we show that in general the long-run average costs are non-convex with respect to the policy parameters. In the numerical study parameter settings are identified where the system behaves like a PUSH system (i.e. everything arrives to the upstream is immediately pushed to downstream stages). Furthermore, the impact of the simplifying fixed cost assumption is investigated when optimal and non-optimal review periods are used. The results show that when the review periods are optimized, the associated optimal base stock levels ensure that each order results into a positive shipment. In this case, the long-run average costs are convex and the base stock levels can be recursively computed as shown in Van Houtum et al. (2007).

The remainder of the paper is structured as follows. First, relevant literature on serial systems with and without fixed ordering costs is discussed in the next section. Next, we describe the mathematical model of the two-echelon system in detail in Sect. 3. Then, in Sect. 4, we present the analytical results on the optimization problem followed by the results of the numerical study in Sect. 5. The summary of the main results and a discussion of future research directions conclude the paper.

2 Literature review

Closely related to our paper is literature on multi-echelon inventory systems under periodic review. However, in this setting fixed costs are often not considered at all or only implicitly mentioned. For serial systems with no fixed ordering costs orders can be placed at the beginning of each period. For this situation, base stock policies are shown to be optimal for the finite horizon case by Clark and Scarf (1960) and for the infinite horizon case by Federgruen and Zipkin (1984). Fixed order costs are implicitly considered in Van Houtum et al. (2007), because they assume that review periods can be different at different stages, but they are given and have to satisfy the integer-ratio constraint. Under these assumptions they have proven that base-stock policies are optimal with respect to holding and backorder costs and they additionally provide newsboy-type formulas to compute the optimal numerical values for the base stock levels. Fixed costs are explicitly considered in Feng and Rao (2007) where optimal policy parameters, review periods as well as base stock levels, are determined for a serial two-echelon inventory system controlled by a periodic base-stock policy. A computational optimization technique based on simulation and a Golden Section search is presented to compute the optimal policy parameters for Poisson distributed customer demand. In contrast to our paper, they do not provide an analytical expression for the average fixed ordering cost per period. Furthermore we consider a more general demand model.

Periodic review policies in combination with fixed order sizes can also result in (R, s, nQ) policies, where R is the review period, s represents the reorder level, Q is the batch size, and n is an integer. By generalizing the work of Chen (2000) and Van Houtum et al. (2007), Chao and Zhou (2009) show the optimality of (R, s, nQ) policies for a multi-echelon serial system with batch ordering and nested replenishment intervals. However, their objective function only includes shortage and holding costs.

Furthermore, (R, s, nQ) policies are studied for a serial system by Shang and Zhou (2009). They propose a heuristic to compute policy parameters minimizing the average system-wide cost including fixed costs for ordering and for each review of the inventory status. Later on, Shang and Zhou (2010) develop another heuristic for the same system that outperforms the previous one. Shang et al. (2010) analyze a serial system with fixed ordering costs, incurred at each order moment. They compare an installation stock (R, S) policy with an echelon (R, S) policy and a continuous review (s, nQ) policy. Note that echelon stock policies are based on system-wide state information and final customer demand, while installation stock policies are

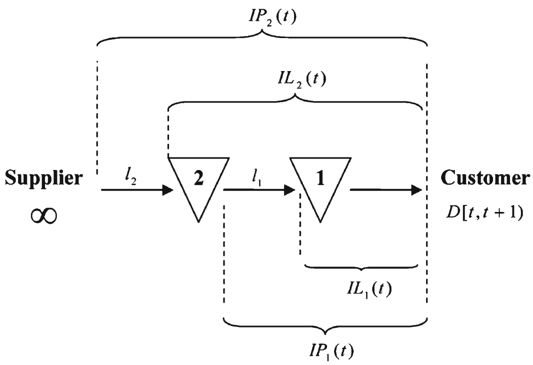

Fig. 1 Illustration of the inventory model

based on local (stockpoint) state information and immediate successors' demand. Here, the echelon (R, S) policy reflects the fact that real-time demand information is shared between stockpoints and the (s, nQ) policy allows flexible deliveries of replenishment orders between stockpoints. The results of their numerical study show that the (s, nQ) policy yields lower system-wide cost than the echelon stock (R, S) policy. Similar results have been provided before for single stockpoint models (see Hadley and Whitin 1963; Rao 2003) and multi-location models (see Cachon 2001; Gurbuz et al. 2007).

We would like to highlight once more, that in most papers in literature (see Shang and Zhou 2009, 2010; Shang et al. 2010) the fixed ordering costs are assumed to be incurred at each order moment and upstream availability of material is not considered, which means that fixed ordering costs are overestimated. In order to investigate this effect, we study a model where fixed costs are only incurred when material is shipped.

3 A serial two-echelon (R, S) model with fixed ordering cost

We consider a single item two-echelon serial inventory system where the stockpoints are labeled, starting from the downstream stockpoint to the upstream stockpoint as 1 and 2 (see also Fig. 1). Stockpoint 2 receives materials from an external supplier with infinite material availability, while stockpoint 1 replenishes its inventory from stockpoint 2. This order quantity is restricted by the stock on-hand at stockpoint 2 such that it may happen that an order from stockpoint 1 can only be partially fulfilled. Even the situation may occur that no material is available at stockpoint 2, so that no shipment results from the order.

Time is divided into periods of equal length and the planning horizon is infinite. We want to make a clear distinction between a "period" and a "review period". Without loss of generality, each period is assumed to have length 1 and periods are numbered as $\{0, 1, 2, \ldots\}$. On the other hand, the review period R_n of stockpoint n, can be composed of multiple periods where at the beginning of a review period the inventory is reviewed and orders can be placed. We assume that stockpoint 2's review period is an integer multiple of stockpoint 1's review period, resulting in the condition $R_2 = rR_1$ with $r = 1, 2, 3, \ldots$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of positive integers. This assumption, also called integer-ratio constraint, is common for such kind of systems, because it facilitates synchronization of ordering and it accounts for the fact fixed costs increase as we move to the upper echelon.

Stochastic customer demand is satisfied from stockpoint 1 and unsatisfied demand is backlogged. Demand in each period is independent and identically distributed with expected value μ, variance σ^2, and coefficient of variation CV where $D[t, t+1)$ represents the demand during period t and is nonnegative. Cumulative demand occurring during a time interval $[t_1, t_2)$ with $(0 \leq t_1 < t_2)$ is denoted by $D[t_1, t_2)$.

The system is under central control and global information is used to determine replenishment quantities, so an echelon stock policy is applied. The concept of echelon stocks is commonly used in multi-echelon inventory systems literature since Clark and Scarf (1960). Also, our model is closely related to the models studied in Van Houtum et al. (2007) and Feng and Rao (2007), where a system-wide echelon stock perspective is used. Thus, we follow the same concepts in our model to be comparable to these papers in terms of analytical results.

The echelon stock of a stockpoint is defined as all stock at this stockpoint plus in transit to or on hand at any of its downstream stockpoints minus the backorders at its downstream stockpoints. We define the echelon inventory position of a stockpoint as its echelon stock plus all material in transfer to that stockpoint. Let $IL_n(t)$ and $IP_n(t)$ be the echelon stock and echelon inventory position for stockpoint n at the beginning of a period t after ordering decisions have been made. At each review period, stockpoint n's echelon inventory position is raised to the base stock level S_n, if possible. This policy is called an echelon base stock policy given by the parameters (R_n, S_n).

We use the concept of synchronization for the ordering moments of stockpoints such that each order arrival time of stockpoint 2 coincides with an ordering moment of stockpoint 1. Without loss of generality, we assume that stockpoint 2 places its first order at the beginning of period 0. Then the set of ordering moments of stockpoint 2 can be defined by $T_2 = \{kR_2 | k \in \mathbb{N}_0\}$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is the set of natural numbers including zero. The lead time l_n between placement and arrival of an order for stockpoint n is assumed to be deterministic and it is defined in periods. Therefore, an order placed by stockpoint 2 at period t $(t \in T_2)$ arrives at the beginning of period $t + l_2$. At this time instant stockpoint 1 can place an order. The synchronized ordering moments of stockpoint 1 are given by $T_1 = \{l_2 + kR_1 | k \in \mathbb{N}_0\}$. With this constraint, we guarantee that the arriving orders to stockpoint 2 can be immediately forwarded to stockpoint 1 resulting in lower holding costs.

We assume linear inventory holding and backorder costs. Each unit backlogged at the end of a period is charged a penalty cost p. Each unit in stock at stockpoint 2 at the end of a period is charged a holding cost $h_2 > 0$. The added value of an item held on stock at echelon 1 is denoted as $h_1 \geq 0$. Thus, a unit in stock at stockpoint 1 at the end of a period incurs a holding cost of $h_1 + h_2$. In each period where an order is received, a fixed ordering cost K_n $(n = 1, 2)$ is charged.

Summarizing, there are four main events that may occur during a period:

- Arrival of orders (if scheduled to this period),
- Placement of orders (if the period is a review period),

- Occurrence of demand,
- Incurrence of costs.

The first two events take place at the beginning of the period. If these events are scheduled to the same period, we assume that ordering decisions are made just after receiving the shipment. Holding and penalty costs are incurred at the end of each period while fixed costs are charged after arrival of orders at the beginning of a period. Customer demand may occur at any point in time during a period.

The objective of this research is to determine base stock levels S_n and review periods R_n minimizing the long run average system-wide cost per period. Before we can formulate the total costs in one period, which is simply the summation of holding, backlogging, and fixed ordering costs for each echelon, we have to introduce some notation. Since we may not charge fixed cost at each period, a variable $\delta_n(t)$ is defined for stockpoint n as follows:

$$\delta_n(t) = \begin{cases} 1 & \text{if an order arrives at stockpoint } n \text{at the beginning of the period } t, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $X_n(t)$ be the echelon stock of stockpoint n at the end of period t. Also, for any $x \in \mathbb{R}$, we define the operators "+" and "−" as $x^+ = \max\{0, x\}$ and $x^- = -\min\{0, x\} = \max\{0, -x\}$ such that $x = x^+ - x^-$. Then, the cost at the end of period t can be written as:

$$\begin{aligned}
&\delta_2(t)K_2 + \delta_1(t)K_1 + h_2(X_2(t) - X_1(t)) + (h_1 + h_2)X_1(t)^+ + pX_1(t)^- \\
&\quad = \delta_2(t)K_2 + \delta_1(t)K_1 + h_2(X_2(t) - X_1(t)) + (h_1 + h_2)(X_1(t) + X_1(t)^-) + pX_1(t)^- \\
&\quad = \delta_2(t)K_2 + \delta_1(t)K_1 + h_2X_2(t) + h_1X_1(t) + (p + h_1 + h_2)X_1(t)^-.
\end{aligned} \tag{2}$$

This one period cost can be decomposed into two parts such that each part represents costs attached to each echelon. Let $C_n(t)$ denote the costs attached to echelon n for period t. Then, the total costs at period t can be separated in:

$$C_1(t) = \delta_1(t)K_1 + h_1X_1(t) + (p + h_1 + h_2)X_1(t)^-, \tag{3}$$

$$C_2(t) = \delta_2(t)K_2 + h_2X_2(t). \tag{4}$$

Our aim is to minimize the long-run average expected cost per period, which is defined as G, as a function of the decision variables R_1, R_2, S_1, S_2:

$$G(R_1, R_2, S_1, S_2) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} (C_1(t) + C_2(t)). \tag{5}$$

We obtain the following optimization problem (P) to be studied:

$$\begin{aligned}
(P):\ \text{Min}\ \ & G(R_1, R_2, S_1, S_2) \\
\text{s.t.}\ \ & R_2 = rR_1, \quad R_1, R_2, r \in \mathbb{N}, \\
& S_1, S_2 \geq 0.
\end{aligned} \tag{6}$$

Table 1 Summary of model parameters and variables

Parameters and variables related to stock information	
$IL_n(t)$	Echelon stock for stockpoint n at the beginning of period t after ordering decisions
$IP_n(t)$	Echelon inventory position for stockpoint n at the beginning of period t after ordering decisions
S_n	Echelon base stock level for stockpoint n
$X_n(t)$	Echelon stock of stockpoint n at the end of period t
x^+	Maximum of 0 and x for any variable x
x^-	Maximum of 0 and $-x$ for any variable x
Parameters and variables related to timing of orders	
l_n	Lead time for arrival of orders for stockpoint n
R_n	Replenishment interval of stockpoint n
r	Number of times stockpoint 1 orders per order of stockpoint 2
$\delta_n(t)$	Binary variable indicating the arrival of an order at stockpoint n at the beginning of period t
Parameters and variables related to the demand	
$D[t_1, t_2)$	Cumulative customer demand between the time interval $[t_1, t_2)$
μ	Expected value of demand per period
σ	Standard deviation of demand per period
CV	Coefficient of variation of demand per period
Parameters and variables related to the costs	
$C_n(t)$	Costs attached to echelon n at the end of period t
h_n	Added inventory holding cost per period for each unit at echelon n
K_n	Fixed ordering cost for stockpoint n
p	Penalty cost for unit backlog

A summary of all problem parameters and variables used in the paper is presented in Table 1.

4 Analysis

In order to solve problem P, we need an analytical expression for the objective function $G(R_1, R_2, S_1, S_2)$. We follow the approach in Van Houtum et al. (2007) and define replenishment cycles for each stage of the system before we derive the expected average cost and necessary conditions for optimality.

4.1 Replenishment cycles

At every R_2 period, an order is shipped from the external supplier to the system and these orders subsequently determine the stock levels at the downstream stages and affect the costs of the system. Assume now that stockpoint 2 places an order at the beginning of period $t_0 \in T_2$. Stockpoint 2 is able to order up to S_2 since it has access to an infinite capacity supplier. Thus, $IP_2(t_0) = S_2$. This order is received at the

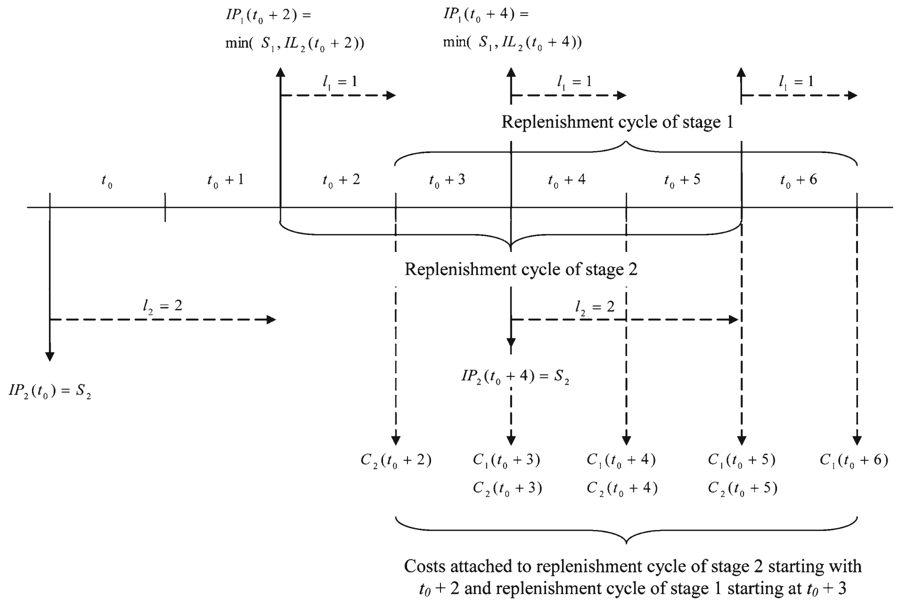

Fig. 2 The replenishment cycles and related costs for $R_2 = 4$, $R_1 = 2$, $l_2 = 2$, $l_1 = 1$

beginning of period $t_0 + l_2$ and the present stock at level 2 is used by stockpoint 1 until the next replenishment order arrives at $t_0 + l_2 + R_2$. Therefore, the order placed at period t_0 by stockpoint 2 affects the holding and ordering costs of echelon 2 at the end of the periods $t_0 + l_2 + k$ for $k = 0, \ldots, R_2 - 1$. Due to the synchronization of order moments, stockpoint 1 is allowed to place an order at the beginning of period $t_0 + l_2$ just after the ordered items arrive to stockpoint 2. Ideally, stockpoint 1 aims to order up to S_1 but this decision is restricted by the echelon stock at stockpoint 2 which is $S_2 - D[t_0, t_0 + l_2 + 1)$ at the beginning of period $t_0 + l_2$. During the time interval $[t_0 + l_2, t_0 + l_2 + R_2)$ stockpoint 1 places r consecutive orders at periods $t_0 + l_2 + i R_1$ for $i = 0, \ldots, r - 1$. These orders arrive after l_1 time units and affect the costs at the end of the periods $t_0 + l_2 + l_1 + i R_1 + j$ for $j = 0, \ldots, R_1 - 1$ and $i = 0, \ldots, r - 1$. For an illustration we refer to Fig. 2.

Based on these observations, we define a "replenishment cycle of stage 2" as the time interval which starts with an order arrival from the external warehouse and ends just before the next arrival. During this cycle, stockpoint 2 can receive maximal one order. The items that arrive to stockpoint 2 will reach stockpoint 1 at earliest l_1 periods later. We define a "replenishment cycle of stage 1" as the time interval which starts in period $t_0 + l_2 + l_1 + k R_2$, and ends just before the period $t_0 + l_2 + l_1 + (k+1) R_2$, for each $k \in \mathbb{N}_0$. Thus, replenishment cycle of stage 1 also has length R_2 and stockpoint 1 may receive new items r times during its cycle. This means, the concept of shifted replenishment cycles is applied. What stockpoint 1 can order during its replenishment cycle is limited to what is available at stockpoint 2 at the beginning of periods $t_0 + l_2 + k R_2$, where $k \in \mathbb{N}_0$. The moments of ordering decisions, the replenishment cycles and the attached costs are illustrated in Fig. 2 for $R_2 = 4$, $R_1 = 2$, $l_2 = 2$, $l_1 = 1$.

In general, the total costs attached to the replenishment cycles of stage 2 and stage 1, triggered by the ordering decision of stockpoint 2 at the beginning of period t_0, are given as

$$\sum_{k=0}^{R_2-1} C_2(t_0 + l_2 + k) + \sum_{i=0}^{r-1} \sum_{j=0}^{R_1-1} C_1(t_0 + l_1 + l_2 + iR_1 + j). \quad (7)$$

4.2 Derivation of the objective function

The cyclic pattern explained in Sect. 4.1 repeats itself throughout the planning horizon. In the long-run, this inventory model can be considered as a renewal reward process (see Tijms (1986)), such that for each stockpoint n a replenishment cycle of stage n represents a renewal cycle and the reward equals the total costs attached to this cycle. Then, for any $t_0 \in T_2$ the long-run average cost per period will be equivalent to the expected cost of related replenishment cycles of both stages divided by the cycle length R_2.

$$\begin{aligned} &G(R_1, R_2, S_1, S_2) \\ &\quad = \frac{1}{R_2} \sum_{k=0}^{R_2-1} E[C_2(t_0 + l_2 + k)] + \frac{1}{R_2} \sum_{i=0}^{r-1} \sum_{j=0}^{R_1-1} E[C_1(t_0+l_1+l_2+iR_1+j)]. \end{aligned} \quad (8)$$

The analysis of the objective function of problem P boils down to the analysis of the expected echelon stocks and expected order frequencies during the replenishment cycles of both stockpoints.

We first analyze the expected echelon stocks during the replenishment cycle of stockpoint 2. We know that the echelon inventory position of stockpoint 2 is raised up to S_2 at the beginning of period t_0 and the order arrives at the beginning of period $t_0 + l_2$. The echelon stock at the end of period $t_0 + l_2 + k$ is the difference between S_2 and the demand during $l_2 + k + 1$ periods where $k = 0, \ldots, R_2 - 1$. Thus, the expected value of $X_2(t_0 + l_2 + k)$ is given by

$$\begin{aligned} E[X_2(t_0 + l_2 + k)] &= E[S_2 - D[t_0, t_0 + l_2 + k + 1)] \\ &= S_2 - (l_2 + k + 1)\mu. \end{aligned} \quad (9)$$

Stockpoint 1 may not be able to raise its echelon inventory position up to S_1 if there is not enough stock available at the upper stockpoint. In this case, we say that there is a shortfall in the echelon inventory position of stockpoint 1. We define the shortfall as the difference between the target value S_1 and the actual inventory position of stockpoint 1 at the beginning of a review period t after placement of orders. Denoting the shortfall with $B_1(t)$ the following holds at ordering moments $t_0 + l_2 + iR_1$ for $i = 0, \ldots, r-1$.

$$\begin{aligned} B_1(t_0 + l_2 + iR_1) &= (S_1 - IL_2(t_0 + l_2 + iR_1))^+ \\ &= (D[t_0, t_0 + l_2 + iR_1) - (S_2 - S_1))^+. \end{aligned} \quad (10)$$

The shortfall determines the echelon inventory position of stockpoint 1 after ordering given as $IP_1(t_0+l_2+iR_1) = \min\{S_1, IL_2(t_0+l_2+iR_1)\} = S_1 - B_1(t_0+l_2+iR_1)$. Using the shortfall notation, the expected echelon stock at the end of each period turns out to be

$$\begin{aligned}
&E[X_1(t_0+l_1+l_2+iR_1+j)] \\
&\quad = E[S_1 - B_1(t_0+l_2+iR_1) - D[t_0+l_2+iR_1, t_0+l_1+l_2+iR_1+j+1)] \\
&\quad = S_1 - (l_1+j+1)\mu - E[B_1(t_0+l_2+iR_1)]. \qquad (11)
\end{aligned}$$

The amount of backorders is represented by $X_1(t)^-$ at the end of period t, which can be interpreted as the shortfall of the customer. Combined with the definition of shortfall at stockpoint 1 the amount of backorders for periods $t_0+l_1+l_2+kR_1+j$ with $i = 0, \dots, R_2-1$ and $j = 0, \dots, R_1-1$ becomes:

$$\begin{aligned}
&X_1(t_0+l_1+l_2+iR_1+j)^- \\
&\quad = (S_1 - B_1(t_0+l_2+iR_1) - D[t_0+l_2+iR_1, t_0+l_1+l_2+iR_1+j+1))^- \\
&\quad = (B_1(t_0+l_2+iR_1) + D[t_0+l_2+iR_1, t_0+l_1+l_2+iR_1+j+1) - S_1)^+. \\
&\qquad (12)
\end{aligned}$$

In order to analyze the expected fixed costs, we should know under which conditions a stockpoint cannot place an order. According to the (R, S)-policy, a stockpoint should place an order at each review period if its inventory position is below S. If there is no customer demand between two ordering moments of a stockpoint, the inventory position will stay at the same level, and, as a result, there will be no ordering. If there is positive customer demand, an order cannot be placed when there is no stock available at the upper stockpoint.

Since stockpoint 2 has a connection with an infinite supply depot, an order is placed, if the customer demand during the previous R_2 periods was positive. The expected order frequency of stockpoint 2 given for $k = 0, \dots, R_2-1$ is obtained as

$$E[\delta_2(t_0+l_2+k)] = \begin{cases} P\{D[t_0-R_2, t_0) > 0\} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (13)$$

The expected ordering decision of stockpoint 1 depends on the amount of stock available at the second stockpoint as well as on customer demand during previous periods. It is clear that the order that arrived at stockpoint 2 at the beginning of period t_0+l_2 will be used by stockpoint 1 till the end of r consecutive replenishment intervals of 1. If there is a shortfall of stockpoint 1 at the previous review period in a replenishment cycle, this means that no stock is left at stockpoint 2 for the remaining replenishment decisions of stockpoint 1. Therefore, the probability of receiving a shipment at stockpoint 1 depends on the previous arrivals of shipments, and it decreases towards the following review periods during the replenishment cycle. Theorem 1 gives the expected order frequencies for stockpoint 1. The proof is provided in the Appendix.

Theorem 1 *The expected number of orders of stockpoint* 1 *during a replenishment cycle of stage 1 starting with period* $t_0 + l_1 + l_2$ *is given as*

$$E[\delta_1(t_0 + l_1 + l_2 + iR_1 + j)] = \begin{cases} P_1 + P_2 + P_3(0), & i = 0, j = 0, \\ P_3(i), & i = 1, \ldots, r-1, j = 0, \\ 0, & i = 0, \ldots, r-1, \\ & j = 1, \ldots, R_1 - 1, \end{cases}$$

where P_1*,* P_2*, and* $P_3(i)$ *are*

$$\begin{aligned} P_1 &= P\{D[t_0 - R_2, t_0 + l_2 - R_1) > S_2 - S_1, D[t_0 - R_2, t_0) > 0\}, \\ P_2 &= P\{D[t_0 - R_2, t_0 + l_2 - R_1) = S_2 - S_1, D[t_0 - R_2, t_0) > 0, \\ &\quad D[t_0 + l_2 - R_1, t_0 + l_2) > 0\}, \\ P_3(i) &= P\{D[t_0, t_0 + l_2 + (i-1)R_1) < S_2 - S_1\} \\ &\quad \cdot P\{D[t_0 + l_2 + (i-1)R_1, t_0 + l_2 + iR_1) > 0\}. \end{aligned}$$

The main idea behind the proof of this theorem is considering the cases, where the following three conditions are fulfilled just before stockpoint 1 places an order:

- The period should be a review period of stockpoint 1,
- There should be positive physical stock at stockpoint 2,
- Stockpoint 1 should have a positive shortfall.

The expressions P_1, P_2, and $P_3(i)$ represent the probabilities that stockpoint 1 places an order if at the end of the previous review period of stockpoint 1:

1. There is no stock left at stockpoint 2 and there is shortfall at stockpoint 1 (P_1),
2. There is no stock left at stockpoint 2 and there is no shortfall at stockpoint 1 (P_2),
3. There is stock left at stockpoint 2 ($P_3(i)$).

Obviously, under cases 1 and 2, there must be an order arrival to stockpoint 2 before stockpoint 1 can place an order. An order arrival to stockpoint 2 is only possible at the beginning of period $t_0 + l_2$, which is the first review period of stockpoint 1 related to replenishment cycle of stage 1 starting with period $t_0 + l_1 + l_2$.

It can be observed that the expected number of orders of stockpoint 1 during a replenishment cycle depends on the difference of S_2 and S_1. If this difference gets infinitely large, stockpoint 1 will be able to order up to S_1 at the beginning of each review period since it has access to an infinite capacity supplier. In the other extreme case, when the base stock levels of both stockpoints are the same, all material arriving to stockpoint 2 is pushed through the system to reach the target base stock level for stockpoint 1. Thus, stockpoint 1 orders all the available stock at stockpoint 2 at its first review period. As a result, stockpoint 1 can receive only one order during its replenishment cycle independent of the number of possible order arrival moments.

Now we are in a position to reformulate the average expected cost per period by using Eqs. (9–13) in (8). We obtain:

$$
\begin{aligned}
G(R_1, R_2, S_1, S_2) = {} & h_2(S_2 - (l_2 + \frac{1}{2}(R_2 + 1))\mu) + \frac{K_2}{R_2} P\{D[t_0 - R_2, t_0) > 0\} \\
& + h_1\left(S_1 - (l_1 + \frac{1}{2}(R_1 + 1))\mu - \frac{R_1}{R_2} \sum_{i=0}^{r-1} E[B_1(t_0 + l_2 + iR_1)] \right) \\
& + (p + h_1 + h_2)\frac{1}{R_2} \sum_{i=0}^{r-1} \sum_{j=0}^{R_1-1} E[X_1(t_0 + l_1 + l_2 + iR_1 + j)^-] \\
& + \frac{K_1}{R_2} \sum_{i=0}^{r-1} E[\delta_1(t_0 + l_1 + l_2 + iR_1)]. \qquad (14)
\end{aligned}
$$

The derivation of (14) is provided in the Appendix.

As mentioned before, many authors neglect the availability of stock when considering fixed ordering cost and charge fixed costs whenever the inventory position of stockpoint n is below S_n at the beginning of its review period. In this case, the last term in Eq. (14) changes and is only dependent on the probability of having a positive demand $\frac{K_1}{R_2} \sum_{i=0}^{r-1} P\{D[t_0 + l_2 + (i-1)R_1, t_0 + l_2 + iR_1) > 0\}$. The expression related to K_2 is the same, since the external supplier has always enough stock. Later we will discuss the impact of this simplifying assumption.

4.3 Optimization of policy parameters

At first, we determine properties for the optimal base stock levels for given review periods using the partial derivatives of $G(R_1, R_2, S_1, S_2)$ with respect to S_1 and S_2. We get the necessary conditions for optimality as shown in Theorem 2 setting the derivatives equal to zero. The proof of this theorem follows from standard calculus, and is therefore omitted (we also refer to Van Houtum et al. 2007).

Theorem 2 *Let $S_1^*(R_1, R_2)$ and $S_2^*(R_1, R_2)$ be the optimal base stock levels of problem P for given values of R_1 and R_2. Given that period demand is modeled as a continuous random variable with the density function $f(\cdot)$ and the distribution function $F(\cdot)$, $S_1^*(R_1, R_2)$ and $S_2^*(R_1, R_2)$ are solutions of the following equations:*

$$
\frac{1}{R_2} \sum_{i=0}^{r-1} \sum_{j=0}^{R_1-1} P\{X_1(t_0 + l_1 + l_2 + iR_1 + j)^- = 0\} = \frac{p}{p + h_1 + h_2}, \qquad (15)
$$

$$
\frac{1}{R_2}\left(-h_1 R_1 + (p + h_1 + h_2) \sum_{j=0}^{R_1-1} (1 - F_{l_1+j+1}(S_1)) \right)\left(\sum_{i=0}^{r-1} F_{l_2+iR_1}(S_2 - S_1) \right)
$$

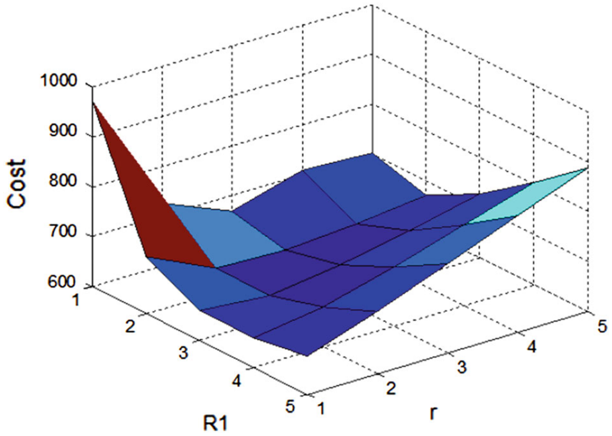

Fig. 3 The cost function with respect to review periods

$$+\frac{K_1}{R_2}\sum_{i=0}^{r-2} f_{l_2+iR_1}(S_2-S_1)=0. \tag{16}$$

where for any positive number x, $f_x(\cdot)$ and $F_x(\cdot)$ represent the x-fold convolution of $f(\cdot)$ and $F(\cdot)$, respectively.

Similar conditions with inequalities can be obtained for discrete demand distributions. The Eq. (15) is dependent on the policy parameters by (10) and (12). This term represents the nonstockout probability per period since it is the sum of probabilities of having no backorders at the end of each period during the replenishment cycle of stage 1 divided by the cycle length. Also, it can be observed that Eq. (15) does not depend on the fixed ordering cost. In fact, this equation is the same as the newsboy equation derived for the model of Van Houtum et al. (2007). On the other hand, Eq. (16) depends on the fixed cost as well as on convolution of the probability density function of the demand. It is easy to see that the term $\frac{K_1}{R_2}\sum_{i=0}^{r-2} f_{l_2+iR_1}(S_2-S_1)$ has a non-monotone behavior in S_2-S_1. As a consequence, the objective function is, in general, not convex. This implies that a multi-start non-linear optimization method should be applied to obtain the globally optimal base stock levels.

With the help of numerical exploration, we concluded that the problem is not convex with respect to the review periods as well. An illustration can be seen in Fig. 3, where R_1 and r are represented as the decision variables for review periods. By definition $R_2 = rR_1$. Thus, we opt for using complete enumeration to compute the optimal numerical values for R_1 and R_2.

5 Numerical study

The aim of our numerical study is twofold. First, we would like to get insights into the structure of optimal ordering moments and an optimal allocation of safety stocks for the system when availability of stock is included in the computation of the average fixed ordering costs. Second, we want to test the impact of the simplifying assumption related to the fixed ordering costs.

For our numerical experiments we consider six different factors: holding cost, service level, coefficient of variation of demand, fixed cost, replenishment intervals, and lead time. Firstly, we fix $h_1 + h_2 = 1$ and change the levels of h_1 and h_2. We assume $h_1 \in \{0.2, 0.5, 0.8\}$ and we vary the penalty cost according to an α service level criterion. This service level is indeed equal to the right-hand side of Eq. (15). Let, α^* be the target service level, then the corresponding penalty cost becomes equivalent to $\frac{\alpha^*}{1-\alpha^*}(h_1 + h_2)$.

We investigate three different values for the target service level $\alpha^* \in \{0.8, 0.9, 0.99\}$ and their corresponding backordering cost p. Further, we take into account three levels for the coefficient of variation of demand, CV $\in \{0.5, 1.0, 1.5\}$. We relate the fixed ordering costs at stockpoint 1 to holding cost and average demand using the EOQ formula. The holding cost at stockpoint 1 is fixed to 1, so we choose two situations where $K_1/\mu = 0.5$ and $K_1/\mu = 2$. To analyze the effect of fixed cost of stockpoint 2, we take two different levels for K_2 for each given value of K_1 ($K_2/K_1 = 1$ and $K_2/K_1 = 2$). Finally, we consider two levels for $l_1 \in \{1, 4\}$ and similar to the setting of fixed ordering costs, two different levels of l_2 are tested proportional to each level of l_1, ($l_2/l_1 \in \{1, 2\}$). All factors and their levels used in the experiments are also presented in the first column of Table 5.

We have conducted a full factorial experiment resulting in 432 problem instances. The optimal policy parameters are computed with the commercial software Matlab where we used a built-in nonlinear optimization tool called "fmincon" to compute the optimal base stock levels S_1^* and S_2^* for given values of R_1 and R_2. The optimal numerical values for the review periods are obtained by exhaustive search. All experiments have been conducted with a desktop computer with 2.8 GHz Intel Core Duo processor and 3.21 GB RAM.

5.1 Structure of an optimal (R, S)-policy

In the first part of our numerical study we have used a Mixed Erlang demand model as a continuous demand distribution due to several reasons. First, Mixed Erlang distributions are very flexible and can represent a large number of different demand patterns Tijms (1986). Second, they are easy to use and there is an exact evaluation procedure available to compute the expected shortfall and the expected number of backorders as described in Van Houtum (2006). In the numerical study, we fix the mean demand to 100 and vary its standard deviation to get the desired coefficient of variation levels.

Part of the numerical results are presented in Table 2 for the situation where the added value at stockpoint 1 is small, i.e. $h_1 = 0.2$.

The first observation is related to the uniqueness of the optimal solution. In many situations there exist multiple optima in terms of review periods. For example, consider the problem instance with $\alpha^* = 0.8$, $CV = 1$ and $K_2 = 400$, where the difference between the base stock levels is strictly zero. No matter how many ordering opportunities stockpoint 1 has, only at the first ordering moment of the replenishment cycle there is stock available to be shipped at stockpoint 2 and only at this point in time

Table 2 Optimal solutions for $h_1 = 0.2, l_1 = l_2 = 1$, and $K_1/\mu = 2$

h_1	α^*	CV	K_2	R_1^*	R_2^*	S_1^*	S_2^*	$S_2^* - S_1^*$	$G(R_1^*, R_2^*, S_1^*, S_2^*)$
0.2	0.8	0.5	200	1	3	507.81	508.17	0.36	405.68
0.2	0.8	0.5	400	1	4	582.40	582.41	0.01	458.55
0.2	0.8	1	200	1	3	564.82	564.82	0.00	549.08
0.2	0.8	1	400	1	4	635.74	635.74	0.00	601.43
0.2	0.8	1.5	200	3	3	625.18	625.18	0.00	712.60
0.2	0.8	1.5	400	1	4	698.84	698.84	0.00	766.58
0.2	0.9	0.5	200	1	3	573.32	573.32	0.00	459.94
0.2	0.9	0.5	400	1	4	657.75	657.76	0.01	519.93
0.2	0.9	1	200	1	3	690.80	690.80	0.00	663.11
0.2	0.9	1	400	1	4	773.55	773.55	0.00	724.73
0.2	0.9	1.5	200	1	3	831.54	831.54	0.00	891.10
0.2	0.9	1.5	400	1	4	912.75	912.75	0.00	956.14
0.2	0.99	0.5	200	3	3	697.07	736.41	39.35	606.13
0.2	0.99	0.5	400	3	3	697.07	736.41	39.35	672.80
0.2	0.99	1	200	1	3	1,050.36	1,050.36	0.00	1,003.08
0.2	0.99	1	400	1	3	1,050.36	1,050.36	0.00	1,069.75
0.2	0.99	1.5	200	1	2	1,274.46	1, 274.46	0.00	1,405.51
0.2	0.99	1.5	400	1	3	1,392.94	1, 392.94	0.00	1,486.32

an order can be executed. Therefore, K_1 is going to be incurred only once during a replenishment cycle of stage 1 independent on the number of possible order moments. Three different combinations of review periods (R_1^*, R_2^*) lead to the same minimal cost: $(1, 4)$, $(2, 4)$, and $(4, 4)$. Note that the same line of thoughts does not hold if upstream unavailability is not considered.

The second observation is also related to the review periods. For a positive difference of the base stock levels the numerical values for the length of the review periods R_1^* and R_2^* are equal and, as we have discussed before, when there is no difference between the optimal base stock levels, we can always find an optimal policy where $R_1^* = R_2^*$ holds. The same results also hold for the case $h_1 = 0.5$. However, as the holding cost at stockpoint 1 increases, it is wiser to keep more stock at upstream stages and let the downstream stages order more frequently. We observe that this effect occurs in many cases when $h_1 = 0.8$ (especially with high α^*) as presented in Table 3.

From the results presented in Table 2, it can also be observed that the differences between the optimal base stock levels are often zero or small. Thus, materials are pushed through the system when h_1 is low and demand variability is large, K_1 is large and lead times are short, and all safety stock is placed close to the customer. Under these parameter settings the system behaves similar to a single stockpoint with lead time $l_1 + l_2$. We have compared the minimal costs of such a system with the cost under an optimal echelon policy for the two stage system for all our problem instances. The relative cost differences Δ are presented in Table 4 and support our findings.

Table 3 Optimal solutions for $h_1 = 0.8$, $l_1 = l_2 = 1$, and $K_1/\mu = 2$

h_1	α^*	CV	K_2	R_1^*	R_2^*	S_1^*	S_2^*	$S_2^* - S_1^*$	$G(R_1^*, R_2^*, S_1^*, S_2^*)$
0.8	0.8	0.5	200	3	3	421.49	525.83	104.34	342.90
0.8	0.8	0.5	400	3	6	421.50	837.92	416.42	380.98
0.8	0.8	1.0	200	3	3	481.58	608.34	126.76	483.53
0.8	0.8	1.0	400	3	6	482.36	950.86	468.50	532.62
0.8	0.8	1.5	200	3	3	550.87	681.38	130.51	649.05
0.8	0.8	1.5	400	4	4	635.00	739.15	104.16	704.40
0.8	0.9	0.5	200	3	3	478.39	595.99	117.60	395.17
0.8	0.9	0.5	400	3	6	478.39	924.98	446.58	435.90
0.8	0.9	1.0	200	3	3	593.74	746.06	152.32	593.99
0.8	0.9	1.0	400	2	6	506.80	1, 140.27	633.47	645.28
0.8	0.9	1.5	200	3	3	742.57	912.57	170.00	823.31
0.8	0.9	1.5	400	2	4	656.54	1, 120.01	463.47	881.48
0.8	0.99	0.5	200	2	4	516.07	913.55	397.48	521.92
0.8	0.99	0.5	400	2	6	516.13	1, 142.75	626.62	566.30
0.8	0.99	1.0	200	2	4	806.36	1, 323.65	517.29	911.94
0.8	0.99	1.0	400	2	4	806.36	1, 323.65	517.29	961.94
0.8	0.99	1.5	200	2	2	1,115.36	1,435.95	320.58	1,306.07
0.8	0.99	1.5	400	2	4	1,116.77	1,763.68	646.91	1,359.63

Table 4 Summary of the results for single stockpoint assumption compared to the optimal solution

		$\%\Delta$	
Factor	Level	Avg.	Max.
h_1	0.2	0.16	1.34
	0.5	1.65	7.06
	0.8	8.75	20.92

5.2 The impact of different cost assumptions

In the second part of the numerical study we have investigated the effect of the assumptions related to the fixed ordering costs. As mentioned before, upstream stock availability is often neglected and to compute the average number of orders during a replenishment cycle only the probability of having positive demand is considered. In this part of the numerical study we have not only considered Mixed Erlang distributed demand, but we have also included a demand distribution where the probability of having no demand in a period is positive. We have chosen a very simple distribution where demand is either zero with probability p_0 or a fixed positive amount (in our study the maximum demand is 5) is demanded with probability $1 - p_0$. For a given coefficient of variation, we set p_0 using $p_0 = \frac{\text{CV}^2}{1+\text{CV}^2}$. This demand distribution represents a situation with non-regular, or even lumpy demand, while a Mixed Erlang distribution is more suitable for regular demand.

Table 5 Impact of the cost assumption

		Regular demand			Lumpy demand		
		$\%\Delta_G$		$\%\Delta_{\text{CPU}}$	$\%\Delta_G$		$\%\Delta_{\text{CPU}}$
Factor	Level	Avg.	Max.	Avg.	Avg.	Max.	Avg.
h_1	0.2	0.00	0.12	79.03	0.01	0.46	12.85
	0.5	0.01	0.27	77.77	0.13	3.25	12.55
	0.8	0.11	0.65	76.10	0.11	3.11	11.93
α^*	0.8	0.04	0.65	77.10	0.15	3.25	13.75
	0.9	0.05	0.36	78.29	0.08	2.72	12.90
	0.99	0.03	0.24	77.50	0.02	2.46	10.68
CV	0.5	0.02	0.52	77.79	0.03	1.20	13.38
	1	0.04	0.65	79.40	0.21	3.25	12.42
	1.5	0.06	0.51	75.70	0.02	0.57	11.53
K_1/μ	0.5	0.05	0.65	77.00	0.02	1.20	10.93
	2	0.03	0.36	78.26	0.14	3.25	9.77
K_2/K_1	1	0.03	0.39	77.82	0.09	3.11	12.18
	2	0.06	0.65	77.44	0.08	3.25	12.70
l_1	1	0.05	0.65	77.91	0.11	3.25	11.62
	4	0.04	0.33	77.35	0.06	2.46	13.27
l_2/l_1	1	0.03	0.47	78.33	0.08	2.79	12.31
	2	0.05	0.65	76.93	0.08	3.25	12.58

For both demand scenarios we have computed the optimal policy parameters $(R_1^*, R_2^*, S_1^*, S_2^*)$ for the situation where stock availability is included for the computation of the average number of orders placed. Also, we computed the optimal policy parameters $(\tilde{R}_1, \tilde{R}_2, \tilde{S}_1, \tilde{S}_2)$ under the simplifying assumption that each order can be shipped. We have compared the actual costs using the relative difference in costs defined as

$$\Delta_G = \frac{G(\tilde{R}_1, \tilde{R}_2, \tilde{S}_1, \tilde{S}_2) - G(R_1^*, R_2^*, S_1^*, S_2^*)}{G(R_1^*, R_2^*, S_1^*, S_2^*)}. \tag{17}$$

In order to measure the impact of the simplification on the computational effort, we consider relative differences in CPU time as follows:

$$\Delta_{\text{CPU}} = \frac{\text{CPU}^* - \text{CPU}_{\text{S}}}{\text{CPU}^*}, \tag{18}$$

where CPU^* and CPU_{S} represent the processing times for finding the optimal solution, and under the simplifying assumption, respectively. The results are shown in Table 5, where in the column with title Avg. the average of all observed relative differences and in the column with title Max. the maximum of all observed relative differences is depicted.

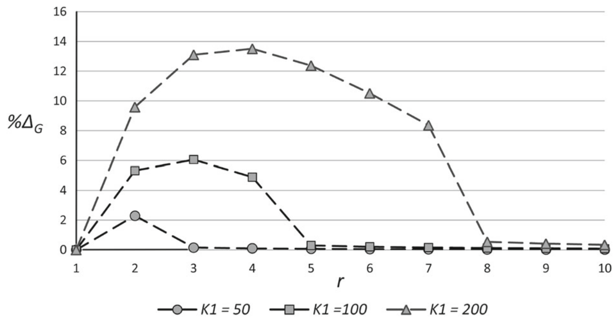

Fig. 4 Non-optimal review periods

It can be seen that the simplifying cost assumption does not have a great impact on the cost performance. When demand is regular nearly no differences can be observed and the differences in the situation with lumpy demand are small. This can be explained by the structure of the optimal policy, where stockpoint 1 only places orders, when there is stock available at stockpoint 2. Thus, in the optimal solution fixed cost is incurred at every review period. As a result the simplifying assumption, works very well with considerably improved CPU performance. For the regular demand case, the average CPU time for finding the optimal solution (solution under simplifying assumption) for one problem is 138.16 (32.77) s. For the lumpy demand case, it takes on average 5.70 (5.04) s to find an optimal solution (solution under simplifying assumption).

As we have mentioned before, periodic review systems are sometimes applied when orders for multiple items have to be coordinated or orders have to be coordinated with transportation schedules. In such situations it may happen, that a hierarchical approach is applied for making decisions, and the review periods are determined first. As a consequence they are exogenous variables in the process of optimizing safety stocks.

We have conducted an experiment where optimal base stock levels are computed for given values of R_2 and we have studied the impact of the cost assumption under Mixed Erlang demand. The relative difference Δ_G is depicted in Fig. 4 for $h_1 = h_2 = 0.5$, $\alpha^* = 0.9$, CV$= 1$, $K_2/K_1 = 2$, $l_1 = l_2 = 1$, and $R_1 = 1$.

As can be seen in Fig. 4, in case of non-optimal review periods, the simplifying assumption for the fixed ordering cost can have a large impact on the cost performance of the obtained policy. Depending on the size of the fixed cost there can be significant differences. We can conclude that under limited flexibility for the review periods, it is crucial to take into account upstream availability when optimizing safety stocks.

6 Summary and outlook

In this paper, we have studied a serial two-echelon inventory system under periodic review and central control. The focus of this research was on the fixed cost related to an order. We have derived an exact expression for the average cost and have provided necessary conditions for the optimal base stock levels. Our model enabled us to investigate the impact of a simplifying assumption related to the fixed ordering cost, which is often the basis of an analysis of these kind of systems. Our numerical study supports the conjecture that the assumption is justified when all policy parameters can be optimized, review periods as well as base stock levels. However, if the ordering moments are constrained by other factors it is crucial to consider upstream availability of stock for the optimization of safety stocks. Moreover, it turns out that for low added holding cost at stockpoint 1 materials should be pushed through the system.

Our study can be seen as a first step, because real-life supply chain structures are usually more complex. Thus, in a next step divergent and convergent or even more general multi-echelon systems should be investigated. Another future research direction can be including other limiting factors such as capacity constraints.

Appendix

Derivation of Eq. (14)

We first derive the expected cost per period attached to replenishment cycle of stage 2. In Eq. (8) this cost is given as:

$$\begin{aligned}
&\frac{1}{R_2}\sum_{k=0}^{R_2-1} E[C_2(t_0+l_2+k)] \\
&= \frac{1}{R_2}\sum_{k=0}^{R_2-1} E[\delta_2(t_0+l_2+k)K_2 + h_2 X_2(t_0+l_2+k)] \\
&= \frac{K_2}{R_2} P\{D[t_0-R_2,t_0) > 0\} + h_2 \frac{1}{R_2}\sum_{k=0}^{R_2-1}(S_2-(l_2+k+1)) \\
&= \frac{K_2}{R_2} P\{D[t_0-R_2,t_0) > 0\} + h_2\left(S_2-\left(l_2+\frac{1}{2}(R_2+1)\right)\mu\right). \qquad (19)
\end{aligned}$$

Similarly, the cost per period attached to replenishment cycle of stage 1 is

$$\begin{aligned}
&\frac{1}{R_2}\sum_{i=0}^{r-1}\sum_{j=0}^{R_1-1} E[C_1(t_0+l_1+l_2+iR_1+j)] \\
&= \frac{1}{R_2}\sum_{i=0}^{r-1}\sum_{j=0}^{R_1-1} E[\delta_1(t_0+l_1+l_2+iR_1+j)K_1 + h_1 X_1(t_0+l_1+l_2+iR_1+j) \\
&\quad +(p+h_1+h_2)X_1(t_0+l_1+l_2+iR_1+j)^-]
\end{aligned}$$

$$
\begin{aligned}
&= h_1(S_1 - (l_1 + \frac{1}{2}(R_1+1))\mu - \frac{R_1}{R_2}\sum_{i=0}^{r-1} E[B_1(t_0 + l_2 + iR_1)]) \\
&+ (p + h_1 + h_2)\frac{1}{R_2}\sum_{i=0}^{r-1}\sum_{j=0}^{R_1-1} E[X_1(t_0 + l_1 + l_2 + iR_1 + j)^-] \\
&+ \frac{K_1}{R_2}\sum_{i=0}^{r-1} E[\delta_1(t_0 + l_1 + l_2 + iR_1)].
\end{aligned} \tag{20}
$$

The summation of (19) and (20) results in (14).

Proof of Theorem 1

Stockpoint 1 will place an order at period t if the following conditions are met:

1. t is a review period for stockpoint 1,
2. Inventory position of stockpoint 1 before placing the order at the beginning of period t is below S_1,
3. The physical stock level at stockpoint 2 is positive at the beginning of period t before stockpoint 1 places an order.

Using these principles we prove each condition separately.

- Case 1: $i = 0, \ldots, r-1$ and $j = 1, \ldots, R_1 - 1$

The periods $t_0 + l_2 + iR_1 + j$ for $i = 0, \ldots, r-1$ and $j = 1, \ldots, R_1 - 1$ are not review periods of stockpoint 1, so there will be no arrival of orders at the beginning of periods $t_0 + l_1 + l_2 + iR_1 + j$ when $j \neq 0$. Thus, $\delta_1(t_0 + l_1 + l_2 + iR_1 + j) = 0$.

- Case 2: $i = 1, \ldots, r-1$ and $j = 0$

The installation stock level of 2 directly affects the decision of ordering of stockpoint 1. It changes only when there is an order arrival from the supplier, or items are ordered from stockpoint 1. We define the installation stock level of stockpoint 2 as $IS_2(t)$ at the beginning of period t, after arrival of orders to stockpoint 2 and after stockpoint 1 places an order. It is equal to the difference between the echelon stock of stockpoint 2 and the inventory position of stockpoint 1 at period t. Throughout the review periods $t_0 + l_2 + iR_1$ such that $i = 1, \ldots, r-1$, there are no order arrivals to stockpoint 2. Also, there will be no shipments from stockpoint 2 to 1 in between two review periods of stockpoint 1. Thus, the installation stock level at stockpoint 2 at period $t_0 + l_2 + iR_1$ before stockpoint 1 places an order will be equal to $IS_2(t_0 + l_2 + (i-1)R_1)$.

$$
\begin{aligned}
IS_2(t_0 + l_2 + (i-1)R_1) &= IL_2(t_0 + l_2 + (i-1)R_1) - IP_1(t_0 + l_2 + (i-1)R_1) \\
&= S_2 - S_1 - D[t_0, t_0 + l_2 + (i-1)R_1) + [D[t_0, t_0 + l_2 + (i-1)R_1) - (S_2 - S_1)]^-.
\end{aligned}
$$

There are two possible outcomes for the installation stock level depending on the amount of demand during the time interval $[t_0, t_0 + l_2 + (i-1)R_1)$:

1. If $D[t_0, t_0 + l_2 + (i-1)R_1) \geq S_2 - S_1$, then $IS_2(t_0 + l_2 + (i-1)R_1) = 0$,
2. If $D[t_0, t_0 + l_2 + (i-1)R_1) < S_2 - S_1$, then $IS_2(t_0 + l_2 + (i-1)R_1) > 0$.

An order can be placed by stockpoint 1 at the beginning of period $t_0 + l_2 + iR_1$, only when $IS_2(t_0 + l_2 + (i-1)R_1) > 0$. If this is the case, it also means that stockpoint 1 has raised its inventory position up to S_1 at period $t_0 + l_2 + (i-1)R_1$. Then, if customer demand during last R_1 periods is positive, stockpoint 1 will place an order at the beginning of period $t_0 + l_1 + l_2 + iR_1$. Thus, the expected value of $\delta_1(t_0 + l_1 + l_2 + iR_1)$ for $i = 1, \ldots, r-1$ is $P\{D[t_0, t_0 + l_2 + (i-1)R_1) < S_2 - S_1\}P\{D[t_0 + l_2 + (i-1)R_1, t_0 + l_2 + iR_1) > 0\}$.

- Case 3: $i = 0$ and $j = 0$

The arguments we have followed for Case 2 also hold for this case. On top of that, there can be an order arrival to stockpoint 2 at the beginning of the period $t_0 + l_2$ and $IS_2(t_0 + l_2)$ can be positive. Therefore, we differentiate three situations where stockpoint 1 can place an order.

1. $IS_2(t_0 + l_2 - R_1) > 0$,
2. $IS_2(t_0 + l_2 - R_1) = 0$ and $IP_1(t_0 + l_2 - R_1) < S_1$,
3. $IS_2(t_0 + l_2 - R_1) = 0$ and $IP_1(t_0 + l_2 - R_1) = S_1$.

At the first situation, stockpoint 1 will place an order if $D[t_0 + l_2 - R_1, t_0 + l_2) > 0$. Here, it is not important if stockpoint 2 receives a shipment at $t_0 + l_2$ since it already has positive stock level. The probability for situation 1 is: $P\{D[t_0, t_0 + l_2 - R_1 < S_2 - S_1\}P\{D[t_0 + l_2 - R_1, t_0 + l_2) > 0\}$.

At the second situation, stockpoint 2 has no physical stock left and stockpoint 1 has positive shortfall. Thus, stockpoint 1 will place an order even if there is no demand during $[t_0 + l_2 - R_1, t_0 + l_2)$. Beware that an order can be shipped from stockpoint 2 to stockpoint 1, if there is an order arrival to stockpoint 2 at the beginning of period $t_0 + l_2$. So, for situation 2, we have $P\{D[t_0 - R_2, t_0 + l_2 - R_1) > S_2 - S_1, D[t_0 - R_2, t_0) > 0\}$ as the expected order frequency.

At the last situation, stockpoint 1 will be below its inventory position if $D[t_0 + l_2 - R_1, t_0 + l_2) > 0$ and it can place an order if there is an arrival to stockpoint 2 at $t_0 + l_2$. The combined probability for ordering becomes $P\{D[t_0 - R_2, t_0 + l_2 - R_1) = S_2 - S_1, D[t_0 - R_2, t_0) > 0, D[t_0 + l_2 - R_1, t_0 + l_2) > 0\}$.

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